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L^p-L^q estimates of damped wave equation and their application

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1 Introduction

We consider the large times asymptotics to the Cauchy Problem to the following damped wave equation:

$$\partial_t^2 u - \Delta u + 2a\partial_t u = 0, \quad u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x) \tag{1}$$

for $(t,x) \in (0,\infty) \times \boldsymbol{R}^n$, where $a > 0$ is a constant. Several authors have investicated that the problem (1) has the diffusive structure as $t \to \infty$([1]). We use the function space $L^p = L^p(\boldsymbol{R}^n)$ with norm $||\cdot||_p = ||\cdot||_{L^p}$. $\mathcal{F}f(\xi) = \hat{f}(\xi)$ denotes Fourier transformation of f with respect on x. Using the solution formula to the problem (1), Marcati-Nishihara[8] and Nishihara[15] obtained the following estimate when $n = 1, 3$:

Theorem 1 *Let $1 \le q \le p \le \infty$ and $\epsilon > 0$. Assume that $u_0 \in L^q$, $u_1 \in L^q$. Let u be the solution of the problem* (1), *and the let v be the solution of the problem:*

$$2a\partial_t v - \Delta v = 0, \quad v(0,x) = u_0(x) + u_1(x)/2a, \tag{2}$$

for $(t,x) \in (0,\infty) \times \boldsymbol{R}^n$. Then the estimate

$$\left|\left|u(t,\cdot) - v(t,\cdot) - e^{-at}M(t)(u_0,u_1)\right|\right|_p \le Ct^{-n\delta-1}(||u_0||_q + ||u_1||_q)$$

holds, for $t > 1$, where $\delta = (1/q - 1/p)/2$ and $M(t)(u_0,u_1)$ is the corrected term related to the wave equation:

$$\partial_t^2 W - \Delta W = 0, \quad (t,x) \in (0,\infty) \times \boldsymbol{R}^n. \tag{3}$$

The first aim is to show that the above Marcati-Nishihara type estimates hold for any space dimension n. We apply Fourier analysis to give estimates to the low frequency part and the high frequency part of the solution to the equation (1). Next consider the nonlinear equation:

$$\partial_t^2 u - \Delta u + 2a\partial_t u = f(u), \quad u(0,x) = u_0(x), \ \partial_t u(0,x) = u_1(x) \tag{4}$$

for $(t,x) \in (0,\infty) \times \boldsymbol{R}^n$, where $f(u) = \pm|u|^\sigma u$, $\pm|u|^{\sigma+1}$.
The second aim is to apply the above L^p-L^q estimates to show the existence of small

data time global solution (SG) to (4), when $n \leq 5$, $2/n < \sigma \leq 2/(n-2)$ if $n \geq 3$, and $2/n < \sigma < \infty$ if $n \leq 2$. It is well known that $\sigma = 2/n$ is the Fujita critical exponent.
Several authors have proved the existence of (SG) to the problem (4)). Matsumura[10] has shown existene of (SG) when $f(u)$ is smooth. Kawashima-Nakao-Ono[7] shown existene of (SG) when $4/n < \sigma$. Marcati-Nishihara[8] and Nishihara[15] applied their L^p-L^q estimates to prove the existence of (SG), provided $2 < \sigma$ when $n = 1$ and $2/3 \leq 2$ when $n = 3$. Todorova-Yordanov[16] have shown the existence of (SG) for general space dimension n, provided that initial data are compactly supported and $2/n < \sigma < 2/(n-2)$ when $n \geq 3$, and $2/n < \sigma$ when $n \leq 2$. Moreover, Todorova-Yordanov[16] and Zhang[17] also have shon that every non trivial solution blows up in finite time, provided that initial data u_0 and u_1 are non-negative and $\sigma \leq 2/n$. Recently, Ikehata[4] and Hayashi-Kaikina-Naumkin[3] have shown the existence of (SG) for general n without the assumption that the initial data are compactly supported, provided that initial data are rapidly decreasing as $|x| \to \infty$ and $2/n < \sigma \leq 2/(n-2)$.

Hayashi-Kaikina-Naumkin[2] and Meier[11] have shown that the problem

$$2a\partial_t V - \Delta V = |V|^\sigma V, \quad V(0,x) = V_0(x), \quad (t,x) \in (0,\infty) \times \boldsymbol{R}^n$$

may admit time global solution even if $\sigma \leq 2/n$, provided that the initial data V_0 are not positive. When initial data are compactly supported and odd with respect one variable, Ikehata-Miyaoka-Nakatake[5] and Ikehata[6] have shown that the problem (4) may admit (SG) even if $\sigma \leq 2/n$.

The third aim is to obtain new L^p-L^q estimate when the initial data are odd, and to show the existence of (SG) to (4), provided that $\sigma_c < \sigma \leq 2/n$, $n \leq 5$. The critical exponent σ_c will be denoted latter.

2 Preliminaries

In this section we state the preliminary results for the proof of L^p-L^q estimates. Let $J_\nu(s)$ be the Bessel function of order ν, and let $\tilde{J}_\nu(s) \equiv J_\nu(s)/s^\nu$.

Lemma 1 *Let ν be not an negative integer, then the followings hold:*

(1) $s\tilde{J}'_\nu(s) = \tilde{J}_{\nu-1}(s) - 2\nu\tilde{J}_\nu(s)$,

(2) $\tilde{J}'_\nu(s) = -s\tilde{J}_{\nu+1}(s)$,

(3) $\tilde{J}_{-1/2}(s) = \sqrt{\frac{\pi}{2}}\cos s$,

(4) *For fixed* Re ν,

$$|\tilde{J}_\nu(s)| \leq Ce^{\pi|\mathrm{Im}\ \nu|}, \qquad (|s| \leq 1),$$

$$J_\nu(s) = Cs^{-1/2}\cos\left(s - \frac{\nu}{2}\pi - \frac{\pi}{4}\right) + O\left(e^{2\pi|\mathrm{Im}\ \nu|}|s|^{-3/2}\right), \quad (|s| \geq 1).$$

(5) $r^2\rho\tilde{J}_{\nu+1}(r\rho) = -\dfrac{\partial}{\partial\rho}\tilde{J}_\nu(r\rho)$.

The following lemmas are well known. See [13] and the references there.

Lemma 2 *Assume that* $\hat{f}(\xi) = g(|\xi|) \in L^1$ *be a radial function, then the equality*

$$f(x) = c\int_0^\infty g(r)r^{n-1}\tilde{J}_{n/2-1}(|x|r)\,dr$$

holds.

Lemma 3 (Hardy-Littlewood-Sobolev) *Let* $1 < q < p < \infty$, $1 - 1/r = 1/q - 1/p$. *Assume that* $|g(x)| \leq A|x|^{-n/r}$, *where* A *is a constant. Then the estimate*

$$||f * g||_p \leq C(p,q)A||f||_q, \qquad f \in L^p$$

holds.

Lemma 4 *Let* $1 \leq p_0, p_1, q_0, q_1 \leq \infty$. *Assume that* $p_0 \neq p_1$, $q_0 \neq q_1$ *and that an operator* T *is bounded from* L^{p_0} *to* L^{q_0} *with norm* M_0, *and that the operator* T *is bounded from* L^{p_1} *to* L^{q_1} *with norm* M_1. *Then, the operator* T *is bounded from* $L^{p(\theta)}$ *to* $L^{q(\theta)}$ *with norm* $M \leq M_0^{1-\theta}M_1^\theta$, *provided that* $0 < \theta < 1$ *and*

$$\frac{1}{p(\theta)} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q(\theta)} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Lemma 5 *Let* $S = \{z = x + iy; 0 < x < 1, y \in \boldsymbol{R}\}$ *be a strip and let* T_z *be an analytic family of linear operators satisfying*

$$||T_{iy}h||_{p_0} \leq A_0N_0(y)||h||_{q_0}, \quad ||T_{1+iy}h||_{p_1} \leq A_1N_1(y)||h||_{q_1}, \quad N_0(0) = N_1(0) = 1$$

where $1 \leq p_j, q_j \leq \infty$ *for* $j = 0, 1$ *and*

$$\sup_{-\infty<y<\infty} e^{-b|y|}\log N_j(y) < \infty$$

for some $b < \pi$. *Then, if* $0 < \theta < 1$, *there is a constant* $C(\theta, b)$ *so that*

$$||T_\theta h||_{p(\theta)} \leq C(\theta,b)A_0^{1-\theta}A_1^\theta||h||_{q(\theta)}$$

for

$$\frac{1}{p(\theta)} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q(\theta)} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Furthermore we may replace $p_1 = \infty$ *with BMO, provided that* $p_0 \neq 1$.

3 L^p-L^q Estimates

In this section we state the L^p-L^q type estimates to the problem (1).
The estimates of low frequency part are as follows.

Theorem 2 (Estimate near $|\xi| = 0$) *Let $1 \leq q \leq p \leq \infty$, $\epsilon > 0$ and $b > 0$ be constants. Assume that $u_i \in L^q$ and supp $\hat{u}_i \subset \{\xi : |\xi| \leq b\}$ for $i = 0, 1$. Let u be the solution of the problem* (1), *and let v be the solution of the problem* (2). *Then, for any multi-index α and for any integer $k \geq 0$, the estimate*

$$\left\|\partial_t^k \partial_x^\alpha \left(u(t,\cdot) - v(t,\cdot)\right)\right\|_p \leq C(1+t)^{-n\delta-j-|\alpha|/2-1+\epsilon} \left(\|u_0\|_q + \|u_1\|_q\right)$$

holds, where $\delta = (1/q - 1/p)/2$. Furthermore, if $1 < q < p < \infty$, $p = q = 2$, or $q = 1, p = \infty$, we may take $\epsilon = 0$.

(sketch) The Fourier transformation of (1) gives

$$\begin{aligned}\hat{u}(t,\xi) &= e^{-at}\cosh t\sqrt{a^2 - |\xi|^2}\hat{u}_0 + e^{-at}\frac{\sinh t\sqrt{a^2-|\xi|^2}}{\sqrt{a^2-|\xi|^2}}(a\hat{u}_0(\xi) + \hat{u}_1(\xi)) \\ &= \exp\left(-\frac{|\xi|^2 t}{2a}\right)\frac{2a\hat{u}_0(\xi) + \hat{u}_1(\xi))}{2a} + \hat{R}_1(t,\xi) + \hat{R}_2(t,\xi),\end{aligned}$$

where

$$\begin{aligned}\hat{R}_1(t,\xi) &= \frac{1}{2}\left\{\exp\left(-at + t\sqrt{a^2-|\xi|^2}\right) - \exp\left(-\frac{|\xi|^2 t}{2a}\right)\right\}\left(\hat{u}_0(\xi) + \frac{a\hat{u}_0(\xi) + \hat{u}_1(\xi)}{\sqrt{a^2-|\xi|^2}}\right) \\ &\equiv g(t,|\xi|)\left(\hat{u}_0(\xi) + \frac{a\hat{u}_0(\xi) + \hat{u}_1(\xi)}{\sqrt{a^2-|\xi|^2}}\right)\end{aligned}$$

and

$$\begin{aligned}\hat{R}_2(t,\xi) &= \exp\left(-\frac{|\xi|^2 t}{2a}\right)\frac{|\xi|^2}{\sqrt{a^2-|\xi|^2}(a+\sqrt{a^2-|\xi|^2})}\frac{a\hat{u}_0(\xi) + \hat{u}_1(\xi)}{2a} \\ &+\frac{1}{2}\exp\left(-at - t\sqrt{a^2-|\xi|^2}\right)\left(\hat{u}_0(\xi) - \frac{a\hat{u}_0(\xi) + \hat{u}_1(\xi)}{\sqrt{a^2-|\xi|^2}}\right).\end{aligned}$$

Choose and fix a radial function χ of class C^∞ with compact support satisfying $\chi(\xi) = 1$ on supp $\hat{u}_0 \bigcup \hat{u}_1$.

Set

$$I(t,x) = \mathcal{F}^{-1}\left(\chi(\xi)g(t,|\xi|)\right),$$

then

$$R_1(t,x) = cI(t,\cdot) *_x \mathcal{F}^{-1}\left(\hat{u}_0(\xi) + \frac{a\hat{u}_0(\xi)+\hat{u}_1(\xi)}{\sqrt{a^2-|\xi|^2}}\right),$$

where $f * g$ is a covolution of f and g.

Lemma 2 shows that

$$(-\Delta)^k I(t,x) = c\int_0^\infty \chi_1(\rho)g(t,\rho)\rho^{n-1+k}\tilde{J}_{n/2-1}(\rho|x|)\,d\rho$$

for any integer $k \geq 0$.
The following proposition gives the estimates $||(-\Delta)^k R_1(t,\cdot)||_p$.

Proposition 1 *Let k be a nonnegative integer, then the following estimates hold for any $t \geq 0$:*

(1) $\sup_x \left|(-\Delta)^k I(t,x)\right| \leq c(1+t)^{-n/2-k-1}$.

(2) $\sup_x (1+|x|)^{n+1/2}\left|(-\Delta)^k I(t,x)\right| \leq c(1+t)^{-3/4-k}$.

To estimate $||R_2(t,\cdot)||_p$, we use standard estimates to the heat equation (2).

Q.E.D.

The estimates of high frequency part are as follows.

Theorem 3 (Estimate near $|\xi| = \infty$) *Let $1 < q \leq p < \infty$. Assume that $u_i \in L^q$ and supp $\hat{u}_i \subset \{\xi : |\xi| \geq 2a\}$ for $i = 0,1$. Let u be the solution of the problem* (1). *Then the estimate*

$$\left\|u(t,\cdot) - e^{-at}M(t)(u_0,u_1)\right\|_p \leq Ce^{-at/2}\left(||u_0||_q + ||u_1||_q\right)$$

holds, where

$$\begin{aligned}\mathcal{F}\{M(t)(u_0,u_1)\} = &\left(\cos t|\xi| \sum_{0\leq k<(n+1)/4} \frac{(-1)^k}{(2k)!}(t\Theta(\xi))^{2k}\right.\\ &\left.+\sin t|\xi| \sum_{0\leq k<(n-1)/4} \frac{(-1)^k}{(2k+1)!}(t\Theta(\xi))^{2k+1}\right)\hat{u}_0(\xi)\\ &+\left(\sin t|\xi| \sum_{0\leq k<(n-1)/4} \frac{(-1)^k}{(2k)!}(t\Theta(\xi))^{2k} - \cos t\xi\right.\\ &\left.\times \sum_{0\leq k<(n-3)/4} \frac{(-1)^k}{(2k+1)!}(t\Theta(\xi))^{2k+1}\right)\frac{a\hat{u}_0(\xi)+\hat{u}_1(\xi)}{\sqrt{|\xi|^2-a^2}},\end{aligned}$$

and $\Theta(\xi) \equiv |\xi| - \sqrt{|\xi|^2 - a^2}$.

(sketch) Fourier transformation of the proble (1) gives

$$\hat{u}(t,\xi) = e^{-at}\cos(t|\xi| - t\Theta(\xi))\hat{u}_0(\xi) + e^{-at}\frac{\sin(t|\xi| - t\Theta(\xi))}{\sqrt{|\xi|^2 - a^2}}(a\hat{u}_0(\xi) + \hat{u}_1(\xi)).$$

MacLaurin expansions of $\cos(t|\xi| - t\Theta)$ and $\sin(t|\xi| - t\Theta)$ with respect to Θ show that we only have to estimate the terms

$$e^{\pm it|\xi|}e^{\pm it\Theta(\xi)}\Theta(\xi)^k\hat{v}(\xi).$$

Since $\Theta(\xi) \doteqdot a^2/2\sqrt{1+|\xi|^2}$ for large $|\xi|$, the solution formula to the wave equation (3), Lemmas 1-2 and 4-5, and Fourier multiplier theory show the desired estimates.

Q.E.D.

Remark 1 Theorems 2-3 *show the character of damped wave equation of* (1). *Let* u *be the solution of* (1), *let* v *be the solution of* (2), *and let* w *be the solution of* (3). *Then*

$$\hat{u}(t,\xi) = \begin{cases} \hat{v}(t,\xi), & \text{for small } \ |\xi|, \\ e^{-at}\hat{w}(t,\xi), & \text{for large } \ |\xi|. \end{cases}$$

The damped wave equation (1) *has the same decay properties as those to the heat equation* (2), *and it has the same regularity properties as those to the wave equation* (3), *though the amplitude of the solution decays exponentially.*

Theorems 2-3 give the following decay estimares.

Theorem 4 (1) *Under the assumptions of* Theorem 2, *the estimates*

$$\left|\left|\partial_t^k\partial_x^\alpha u(t,\cdot)\right|\right|_p \le C(j,\alpha)(1+t)^{-n\delta-k-|\alpha|/2}\left(||u_0||_q + ||u_1||_q\right)$$

holds for $1 \le q \le p \le \infty$.

(2) *Under the assumptions of* Theorem 3, *the estimates*

$$||u(t,\cdot)||_p \le C(p)e^{-at/2}\left(||u_0||_{1,p} + ||u_1||_p\right)$$

hold for $1 < p < \infty$ *when* $1 \le n \le 3$, *and for* $1/2 - 1/2m < 1/p < 1/2 + 1/2m$ *when* $n \ge 4$, *where* $m = [n/2]$.

4 Applications to nonlinear problem

Let $n \le 5$, $2/n < \sigma$ when $n = 1$, 2, and $2/n < \sigma \le 2/(n-2)$ when $3 \le n \le 5$. We show that the nonlinear problem (4) admits small data time global solution (SG).

Theorem 5 *Let $n = 4$, 5, $2/n < \sigma < 1$ and $\sigma \le 2/(n-2)$. Assume that*

$$(u_0, u_1) \in Z_1 \equiv \left(H_2^2 \cap H^1_{1+1/\sigma} \cap H^1_{1+\sigma} \cap L^1\right) \times \left(H_2^1 \cap L^1\right),$$

and set

$$||(u_0,u_1)||_{Z_1} = ||u_0||_{H_2^2} + ||u_0||_{H^1_{1+1/\sigma}} + ||u_0||_{H^1_{1+\sigma}} + ||u_0||_1 + ||u_1||_{H_2^1} + ||u_1||_1 .$$

If $||(u_0,u_1)||_{Z_1}$ is sufficiently small, then the problem (4) *possess a unique solution u in the class*

$$C\left([0,\infty); H_2^2 \cap L^{1+1/\sigma} \cap L^{1+\sigma}\right) \cap\ C^1\left([0,\infty); H_2^1\right) \cap C^2\left([0,\infty); L^2\right),$$

and it satisfies the estimates:

$$||u(t,\cdot)||_p \le C(1+t)^{-(n/2)\cdot(1-1/p)}||(u_0,u_1)||_{Z_1}$$

for $1+\sigma \le p \le 1+1/\sigma$,

$$\left|\left|\partial_t^k\partial_x^\alpha u(t,\cdot)\right|\right|_2 \le C(1+t)^{-n/4-k-|\alpha|/2}||(u_0,u_1)||_{Z_1}$$

for $k+|\alpha| \le 2$, $k \le 1$, and

$$\left|\left|\partial_t^2 u(t,\cdot)\right|\right|_2 \le C(1+t)^{-n/4-n\sigma}||(u_0,u_1)||_{Z_1}.$$

Theorem 6 *Let $n = 3$ and $2/3 < \sigma < 1$. Assume that*

$$(u_0, u_1) \in Z_2 \equiv \left(H^1_{1+1/\sigma} \cap H^1_{1+\sigma} \cap L^1\right) \times \left(L^{1+1/\sigma} \cap L^1\right),$$

and set

$$||(u_0,u_1)||_{Z_2} = ||u_0||_{H^1_{1+1/\sigma}} + ||u_0||_{H^1_{1+\sigma}} + ||u_0||_1 + ||u_1||_{1+1/\sigma} + ||u_1||_1 .$$

If $||(u_0,u_1)||_{Z_2}$ is sufficiently small, then the problem (4) *possess a unique solution u in the class*

$$C\left([0,\infty); H_2^1 \cap L^{1+1/\sigma} \cap L^{1+\sigma}\right) \cap\ C^1\left([0,\infty); L^2\right)$$

and it satisfies the estimates:

$$||u(t,\cdot)||_p \le C(1+t)^{-(3/2)\cdot(1-1/p)}||(u_0,u_1)||_{Z_2}$$

for $1+\sigma \le p \le 1+1/\sigma$,

$$\left|\left|\partial_t^k\partial_x^\alpha u(t,\cdot)\right|\right|_2 \le C(1+t)^{-3/4-k-|\alpha|/2}||(u_0,u_1)||_{Z_2}$$

for $k+|\alpha| \le 1$.

Theorem 7 *Let* $1 \le n \le 4$ *and* $2/n < \sigma$, $\sigma \ge 1$, *and* $\sigma \le 2/(n-2)$ *when* $n \ge 3$. *Assume that*

$$(u_0, u_1) \in Z_3 \equiv \left(H_2^1 \cap L^1\right) \times \left(L^2 \cap L^1\right),$$

and set

$$||(u_0, u_1)||_{Z_3} = ||u_0||_{H_2^1} + ||u_0||_1 + ||u_1||_2 + ||u_1||_1 .$$

If $||(u_0, u_1)||_{Z_3}$ *is sufficiently small, then the problem* (4) *possess a unique solution* u *in the class*

$$C\left([0,\infty); H_2^1\right) \cap C^1\left([0,\infty); L^2\right)$$

and it satisfies the estimates:

$$\left|\left|\partial_t^k \partial_x^\alpha u(t,\cdot)\right|\right|_2 \le C(1+t)^{-n/4-k-|\alpha|/2} ||(u_0, u_1)||_{Z_3}$$

for $k + |\alpha| \le 1$.

Remark 2 Theorems 5-7 *give the following enrgy estimate:*

$$|E(t)| \le C(1+t)^{-n/2-1} ||(u_0, u_1)||_{Z_i}$$

for $i = 1, 2, 3$, *where*

$$E(t) = \frac{1}{2}\left(||\partial_t u(t,\cdot)||_2^2 + ||\nabla u(t,\cdot)||_2^2\right) - \int \frac{|u(t,\cdot)|^{\sigma+2}}{\sigma+2}\,dx.$$

When $1+\sigma \ge 2$, *the several authors have shown the above energy estimates([7], [5]).*

Sketch of the proof of Theorem 5.
Choose and fix a radial function $0 \le \chi_1(\xi) \le 1$ of class C^∞ satisfying

$$\chi_1(\xi) = 1 \quad (|\xi| \le 2a), \qquad \chi_1(\xi) = 0 \quad (|\xi| \ge 3a).$$

We construct the approximate solutions $\{U_j\}_{j=0,1,\ldots}$ to the Cauchy problem (4) as follows: Let $U_{-1} = 0$, and let U_{j+1} be the solution of the Cauchy problem

$$\partial_t^2 U_{j+1} - \Delta U_{j+1} + 2a\partial_t U_{j+1} = f(U_j), \quad (t,x) \in (0,\infty) \times \boldsymbol{R}^n \tag{5}$$

with initial data

$$U_{j+1}(0,x) = u_0(x), \quad aU_{j+1} + 2a\partial_t U_{j+1}(0,x) = u_1(x), \quad x \in \boldsymbol{R}^n \tag{6}$$

for $j \ge -1$. Then the prblem (5)-(6) is equivalent to the following system of the integral equations:

$$v_{j+1}(t,\cdot) = v_0(t,\cdot) + \int_0^t S(t-\tau) f^1(U_j(\tau,\cdot))\,d\tau, \tag{7}$$

$$w_{j+1}(t,\cdot) = w_0(t,\cdot) + \int_0^t S(t-\tau) f^2(U_j(\tau,\cdot))\,d\tau, \tag{8}$$

for $j \geq 0$, where

$$v_j(t,\cdot) = \mathcal{F}^{-1}\left(\chi(\cdot)\hat{U}_j(t,\cdot)\right), \quad w_j(t,\cdot) = \mathcal{F}^{-1}\left((1-\chi(\cdot))\,\hat{U}_j(t,\cdot)\right),$$

$$f^1(U_j(t,\cdot)) = \mathcal{F}^{-1}\left(\chi(\cdot)\hat{f}_j(U_j(t,\cdot))\right),$$

and

$$f^2(U_j(t,\cdot)) = \mathcal{F}^{-1}\left((1-\chi(\cdot))\,\hat{f}_j(U_j(t,\cdot))\right).$$

Then the approximate solutions (v_j, w_j) satisfy:

Lemma 6 *Under the assumptions as ones in* Theorem 5, *it follows that*

$$v_j \in C([0,\infty); L^\infty \cap L^1), \qquad w_j \in C([0,\infty); H_2^2 \cap L^q \cap L^{q'})$$

and

$$U_j \in C([0,\infty); H_2^2) \cap C^1([0,\infty); H_2^1) \cap C^2([0,\infty); L^2)$$

for $j = 0, 1, \cdots$, where $q = 1 + 1/\sigma$ and $q' = 1 + \sigma$.
Moreover, for $j = 0, 1, \cdots$, the approximate solutions (v_j, w_j) satisfy the following estimates:

(1) $||v_j(t,\cdot)||_\infty \leq 2\eta(1+t)^{-n/2}, \quad ||v_j(t,\cdot)||_1 \leq 2\eta,$

(2) $||w_j(t,\cdot)||_q \leq 2\eta(1+t)^{-\beta_1}, \quad ||w_j(t,\cdot)||_{q'} \leq 2\eta(1+t)^{-\beta_2},$
where

$$\beta_1 = \frac{n}{2}\left(1 + \sigma - \frac{1}{q}\right), \quad \beta_2 = \frac{n}{2}\left(1 + \sigma - \frac{1}{q}\right),$$

(3) $||\partial_t^l \partial_x^k v_j(t,\cdot)||_2 \leq 2\eta(1+t)^{-\nu(k,l)}$ *for $k + l \leq 2$, where*

$$\nu(k,l) = \frac{n}{4} + \frac{k}{2} + \min\left(l, \frac{n\sigma}{2}\right)$$

and

$$||\partial_x^k U||_2 = \sum_{\alpha_1+\cdots+\alpha_n=k} ||\partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n} U||_2,$$

(4) $||\partial_t^l \partial_x^k w_j(t,\cdot)||_2 \leq 2\eta(1+t)^{-(n/2)\cdot(\sigma+1/2)-1/2}$ *for $k + l \leq 2$.*

In the above, η is a small constant satisfying $||(u_0, u_1)||_{Z_1} \leq C\eta$.

Lemma 7 *Under the assumptions as ones in* Theorem 5, *the estimate*

$$\sup_{t\geq 0} ||U_{j+1}(t,\cdot) - U_j(t,\cdot)||_2 \leq \frac{1}{2} \sup_{t\geq 0} ||U_j(t,\cdot) - U_{j-1}(t,\cdot)||_2$$

holds for $j \geq 1$.

5 Odd Data Problem

We consider the Cauchy problem (1) with odd initial data. Fix an integer $d \in [1, n]$. Set $x = (x', x'') \in \boldsymbol{R}^n = \boldsymbol{R}^d \times \boldsymbol{R}^{n-d}$, $P(x') \equiv (1 + x_1^2)^{1/2} \cdots (1 + x_d^2)^{1/2}$.

A function $f(x)$ is said to be odd with respect to x' when the equality

$$f(x_1, \cdots, -x_k, \cdots, x_d, x_{d+1}, \cdots, x_n) = -f(x_1, \cdots, x_k, \cdots, x_d, x_{d+1}, \cdots, x_n)$$

holds for any $k \in [1, d]$. The new estimates for the Cauchy problem (1) with odd initial data are as follows;

Theorem 8 (Estimates near $\xi = 0$)
Let $1 \le q \le p \le \infty$. Assume that u_i is odd with respect to x', and $P(x')u_i \in L^q$ $(i = 0, 1)$. Under the assumptions in Theorem 1, *the estimates*

$$\begin{aligned} &\left|\left|P(x')^\theta \partial_t^k \partial_x^\alpha \left(u(t) - v(t)\right)\right|\right|_p \\ &\quad \le C(1+t)^{-n\delta - k - |\alpha|/2 - 1 + \epsilon - (1-\theta)d/2} \left(||P(x')u_0||_q + ||P(x')u_1||_q\right) \end{aligned}$$

hold for $0 \le \theta \le 1$. Furthermore, when $p = q = 2$, $1 < q < p < \infty$ or $p = \infty$ and $q = 1$, we may take $\epsilon = 0$.

Theorem 9 (Estimates near $|\xi| = \infty$)
Let $1 < q \le p < \infty$. Assume that $P(x')u_i \in L^q$ $(i = 0, 1)$. Under the assumptions in Theorem 2, *the estimates*

$$\left|\left|P(x')\left(u(t,\cdot) - e^{-at}M(t)(u_0, u_1)\right)\right|\right|_p \le Ce^{-at/2}\left(||P(x')u_0||_q + ||P(x')u_1||_q\right)$$

hold for $1 < q \le p < \infty$.

Theorem 10 (Time decay)

(1) *Under the assumptions in* Theorem 8, *the estimate*

$$\left|\left|P(x')^\theta \partial_t^k \partial_x^\alpha u(t)\right|\right|_p \le C(1+t)^{-n\delta - k - |\alpha|/2 - (1-\theta)d/2}\left(||P(x')u_0||_q + ||P(x')u_1||_q\right)$$

holds.

(2) *Under the assumptions in* Theorem 9, *the estimate*

$$||P(x')u(t,\cdot)||_p \le C(p)e^{-at/2}\left(||P(x')w_0||_p + ||P(x')w_1||_p\right)$$

hold.

Now we consider the nonlinear problem (4) with odd initial data. Here and after, we assume that initial data u_i are odd with respect to x' for $(i = 0, 1)$.

Theorem 11 *$(n = 4, 5)$ Assume that*
$\sigma_c \equiv 2/(n+d) < \sigma \leq 2/n$, $n + d \leq 6$ and $(u_0, u_1) \in Z_4$,

i.e.,

$$P(x')u_0 \in H_2^2 \cap H^1_{1+1/\sigma} \cap H^1_{1+\sigma} \cap L^1, \quad P(x')u_1 \in H_2^1 \cap L^1,$$

and set

$$||(u_0,u_1)||_{Z_4} = ||P(x')u_0||_{H_2^2} + ||P(x')u_0||_{H^1_{1+1/\sigma}} + ||P(x')u_0||_{H^1_{1+\sigma}} + ||P(x')u_0||_1$$
$$+||P(x')u_1||_{H_2^1} + ||P(x')u_1||_1 .$$

If $||(u_0, u_1)||_{Z_4}$ is sufficiently small, then the problem (4) posesses a unique solution u in class

$$C\left([0,\infty); H_2^2 \cap L^{1+1/\sigma} \cap L^{1+\sigma}\right) \cap C^1\left([0,\infty); H_2^1\right) \cap C^2\left([0,\infty); L^2\right),$$

and u satisfies the estimates:

$$||u(t,\cdot)||_p \leq C(1+t)^{-(n/2)\cdot(1-1/p)-d/2}||(u_0,u_1)||_{Z_4}$$

for $1 + \sigma \leq p \leq 1 + 1/\sigma$,

$$\left\|\partial_t^k \partial_x^\alpha u(t,\cdot)\right\|_2 \leq C(1+t)^{-n/4-\nu(k,\alpha)-d/2}||(u_0,u_1)||_{Z_4},$$

for $k + |\alpha| \leq 1$, $\nu(k,\alpha) = \min\left(k + |\alpha|/2, (n+d)\sigma/2\right)$.

Theorem 12 *$(n = 2, 3)$ Assume that $\sigma_c \equiv 2/(n+d) < \sigma \leq 2/n$, $(u_0, u_1) \in Z_5$, i.e.,*

$$P(x')u_0 \in H^1_{1+1/\sigma} \cap H^1_{1+\sigma} \cap L^1, \quad P(x')u_1 \in L^{1+1/\sigma} \cap L^1,$$

and set

$$||(u_0,u_1)||_{Z_5} = ||P(x')u_0||_{H^1_{1+1/\sigma}} + ||P(x')u_0||_{H^1_{1+\sigma}} + ||P(x')u_0||_1$$
$$+||P(x')u_1||_{1+1/\sigma} + ||P(x')u_1||_1.$$

If $||(u_0,u_1)||_{Z_5}$ is sufficiently small, then the problem (4) admits a unique solution u in the class $C\left([0,\infty); H_2^1 \cap L^{1+1/\sigma} \cap L^{1+\sigma}\right) \cap\ C^1\left([0,\infty); L^2\right)$, and u satisfies the estimates:

$$||u(t,\cdot)||_p \leq C(1+t)^{-(n/2)\cdot(1-1/p)-d/2}||(u_0,u_1)||_{Z_5}$$

for $1 + \sigma \leq p \leq 1 + 1/\sigma$,

$$\left\|\partial_t^k \partial_x^\alpha u(t,\cdot)\right\|_2 \leq C(1+t)^{-n/4-k-|\alpha|/2-d/2}||(u_0,u_1)||_{Z_5}$$

for $k + |\alpha| \leq 1$.

Theorem 13 ($n = 1$) *Assume that* $\sigma_c \equiv 1 < \sigma \leq 2$, $u_0(x)$ *and* $u_1(x)$ *be odd and* $(u_0, u_1) \in Z_6$, *i.e.,*

$$(1+x^2)^{1/2}u_0 \in H_2^1 \cap L^1, \quad (1+x^2)^{1/2}u_1 \in L^2 \cap L^1,$$

and set

$$\begin{aligned}||(u_0,u_1)||_{Z_6} &= ||(1+x^2)^{1/2}u_0||_{H_2^1} + ||(1+x^2)^{1/2}u_0||_1 \\ &+||(1+x^2)^{1/2}u_1||_2 + ||(1+x^2)^{1/2}u_1||_1.\end{aligned}$$

If $||(u_0,u_1)||_{Z_6}$ *is sufficiently small, then the problem* (4) *admits a unique solution*

$$u \in C\left([0,\infty); H_2^1\right) \cap C^1\left([0,\infty); L^2\right),$$

and it satisfies the estimates:

$$\left|\left|\partial_t^j \partial_x^k u(t,\cdot)\right|\right|_2 \leq C(1+t)^{-1/4-j-k/2-1/2}||(u_0,u_1)||_{Z_6}$$

for $j + k \leq 1$.

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